STATED MEETING, WEDNESDAY, NOVEMBER 30, 1859.

JAMES HENTHORN TODD, D. D., President, in the Chair.

The Rev. ROBERT CARMICHAEL, F. T. C. D., read the second part of a paper—

ON CERTAIN METHODS IN THE CALCULUS OF FINITE DIFFERENCES.

SECT. II.—*On the Application of the Calculus of Finite Differences to the Symbolical Reduction of certain Definite Integrals.*

THE theorems principally employed for the deductions of the results contained in the following section are two, fundamental in their character as regards the Calculus of Finite Differences, and easily proved, namely—

$$F(e^{D_x}) \,.\, m^x = F(m) \,.\, m^x, \tag{I.}$$

and

$$F(\Delta) \,.\, m^x = F(m-1) \,.\, m^x, \tag{II.}$$

where F is any algebraic function of the quantity it contains, m any constant, and Δ the ordinary symbol of this Calculus.

1. By the first theorem, if we were required to determine the value of any definite integral of the form

$$\int_{x_1}^{x_2} F(x)\, x^a dx,$$

we see that this integral is instantly reducible to the symbolic shape

$$\int_{x_1}^{x_2} F(e^{D_a})\, x^a dx;$$

and if the quantity a be supposed to be independent of the limits of the integral x_1, x_2, transferring, as is legitimate, the symbolic operator outside the sign of integration, we have, as the symbolic result, simply

$$F(e^{D_a}) \,.\, \left(\frac{x_2^{a+1} - x_1^{a+1}}{a+1}\right),$$

the further evaluation of which will depend upon the particular form of the given function F. The result now obtained admits of ready verification by the substitution for $F(x)$, $F(e^{D_a})$, of their equivalents, derived from the formula

$$F(u) = F(0) + F'(0) \,.\, \frac{u}{1} + F''(0) \,.\, \frac{u^2}{1\,.\,2} + F'''(0) \,.\, \frac{u^3}{1\,.\,2\,.\,3} + \&c.,$$

and its value consists in the circumstance that the question proposed has now become reduced to the mechanical working out of the product of a known operation, upon a simple known subject, that operation proceeding according to a known and practical method.

2. Similarly, if we were required to determine the value of a definite integral of the form,

$$\int_{x_1}^{x_2} F(x-1)\, x^a dx,$$

the value of a being, as before, independent of the limits of the integral, we have, as our symbolic result, in the first instance,

$$F(\Delta_a) \cdot \left(\frac{x_2^{a+1} - x_1^{a+1}}{a+1}\right)$$

(the suffix to the symbol Δ being employed to denote that this symbol is understood to operate only upon a), and the further evaluation, as before, simply depends upon the particular form of the given function F.

3. More generally, if we were required to determine the value of any definite integral of the form

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} F_1(x)\, F_2(y)\, x^a y^\beta dx\, dy,$$

wherein the quantities a, β, are supposed to be independent of the limits and of each other, we see that this integral is equivalent to

$$F_1(e^{D_a})\, F_2(e^{D_\beta}) \cdot \int_{x_1}^{x_2} \int_{y_1}^{y_2} x^a y^\beta dx\, dy,$$

or

$$F_1(e^{D_a})\, F_2(e^{D_\beta}) \cdot \frac{1}{\beta+1} \int_{x_1}^{x_2} x^a \left(y_2^{\beta+1} - y_1^{\beta+1}\right) dx:$$

and if, for simplicity, the equations of the *limiting curves* be written in the form

$$y_2^{\beta+1} = f_2(x), \qquad y_1^{\beta+1} = f_1(x),$$

the final symbolical value of the given integral is

$$F_1(e^{D_a})\, F_2(e^{D_\beta}) \cdot \frac{1}{\beta+1} \{ f_2(e^{D_a}) - f_1(e^{D_a}) \} \cdot \left(\frac{x_2^{a+1} - x_1^{a+1}}{a+1}\right),$$

and the complete evaluation now depends, in general, solely on the particular values of the given functions F_1, F_2, f_1, f_2.

4. Similarly, if it were proposed to determine the value of a definite integral of the form

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} F_1(x-1)\, F_2(y-1)\, x^a y^\beta dx dy,$$

wherein the quantities a, β, are supposed to be independent of the limits and of each other, we see that this integral is equivalent to

$$F_1(\Delta_a) \cdot F_2(\Delta_\beta) \cdot \int_{x_1}^{x_2} \int_{y_1}^{y_2} x^a y^\beta dx dy,$$

or, with the same limiting curves as before,

$$F_1(\Delta_a)\, F_2(\Delta_\beta) \cdot \frac{1}{\beta+1} \{ f_2(e^{D_a}) - f_1(e^{D_a}) \} \cdot \left(\frac{x_2^{a+1} - x_1^{a+1}}{a+1}\right);$$

and, as before, the complete evaluation of this symbolic form depends, in general, solely on the particular values of the given functions, F_1, F_2, f_1, f_2.

5. The results obtained in the preceding articles may be still further generalized. In fact, if the quantities α, β, γ, &c., be supposed to be independent of the limits and of each other, it is evident that we may write the multiple definite integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \ldots F_1(x)\, F_2(y)\, F_3(z) \ldots x^\alpha y^\beta z^\gamma \ldots dx\, dy\, dz \ldots$$

in the form

$$F_1(e^{D\alpha})\, F_2(e^{D\beta})\, F_3(e^{D\gamma}) \ldots \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} x^\alpha y^\beta z^\gamma \ldots dx\, dy\, dz \ldots,$$

and then, solving the simpler integral, proceed to evaluate this result when operated upon by the symbolic factor to the left hand of the expression just obtained.

In the same manner, the multiple definite integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} .. F_1(x-1)\, F_2(y-1)\, F_3(z-1) .. x^\alpha y^\beta z^\gamma .. dx\, dy\, dz ..$$

is seen to be equivalent to

$$F_1(\Delta_\alpha)\, F_2(\Delta_\beta)\, F_3(\Delta_\gamma) .. \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} .. x^\alpha y^\beta z^\gamma .. dx\, dy\, dz ..,$$

and the proposed question is reduced, as before, to the determination of the value of a simpler integral, and the deduction of the expression resulting from the operation upon this value, by the symbolic factor to the left hand of the formula last written down.

6. Since, as is easily seen,

$$e^{D_x + D_y} \,.\, m^x n^y = mn \,.\, m^x n^y,$$

we have, by successive operation,

$$e^{p(D_x + D_y)} \,.\, m^x n^y = (mn)^p \,.\, m^x n^y.$$

Hence, in general, if F be any algebraic function,

$$F(e^{D_x + D_y}) \,.\, m^x n^y = F(mn) \,.\, m^x n^y.$$

By the aid of this theorem, we see that we are readily furnished with means for the reduction and simplification of certain other species of definite integrals.

Thus the same suppositions as before being made with respect to α, β, it appears that the definite integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} F(xy)\, x^\alpha y^\beta\, dx\, dy = F(e^{D_\alpha + D_\beta}) \,.\, \int_{x_1}^{x_2}\int_{y_1}^{y_2} x^\alpha y^\beta\, dx\, dy,$$

and, again, the definite integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} \mathrm{F}\,(\overline{x-1}\,.\,\overline{y-1})\, x^{\alpha} y^{\beta}\, dx\, dy = \mathrm{F}\,(\Delta_{\alpha}\,.\,\Delta_{\beta}) \int_{x_1}^{x_2}\int_{y_1}^{y_2} x^{\alpha} y^{\beta}\, dx\, dy:$$

or, more generally, that the multiple definite integrals

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \ldots \mathrm{F}\,(xyz\ldots)\,.\, x^{\alpha} y^{\beta} z^{\gamma} \ldots dx\, dy\, dz \ldots$$

and

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \ldots \mathrm{F}\,(\overline{x-1}\,.\,\overline{y-1}\,.\,\overline{z-1}\ldots.)\,.\, x^{\alpha} y^{\beta} z^{\gamma} \ldots dx\, dy\, dz \ldots$$

may be reduced, respectively, to the simpler symbolic forms

$$\mathrm{F}\,(e^{D_{\alpha}+D_{\beta}+D_{\gamma}+\ldots})\,.\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \ldots x^{\alpha} y^{\beta} z^{\gamma} \ldots dx\, dy\, dz \ldots$$

and

$$\mathrm{F}\,(\Delta_{\alpha}\,.\,\Delta_{\beta}\,.\,\Delta_{\gamma}\ldots) \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \ldots x^{\alpha} y^{\beta} z^{\gamma} \ldots dx\, dy\, dz \ldots$$

7. Again, it appears from considerations suggested in the last article, that the definite integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} \mathrm{F}\left(\frac{x}{y}\right).\, x^{\alpha} y^{\beta}\, dx\, dy = \mathrm{F}\,(e^{D_{\alpha}-D_{\beta}})\,.\int_{x_1}^{x_2}\int_{y_1}^{y_2} x^{\alpha} y^{\beta}\, dx dy,$$

and the definite integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} \mathrm{F}\left(\frac{x-1}{y-1}\right).\, x^{\alpha} y^{\beta}\, dx dy = \mathrm{F}\left(\frac{\Delta_{\alpha}}{\Delta_{\beta}}\right).\int_{x_1}^{x_2}\int_{y_1}^{y_2} x^{\alpha} y^{\beta}\, dx dy;$$

or, more generally, that the definite integral of the fourth order

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}\int_{w_1}^{w_2} \mathrm{F}\left(\frac{xz}{yw}\right) x^{\alpha} y^{\beta} z^{\gamma} w^{\delta}\; dx\, dy\, dz\, dw$$

may be reduced to the symbolic form

$$\mathrm{F}\,(e^{D_{\alpha}+D_{\gamma}-(D_{\beta}+D_{\delta})})\,.\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}\int_{w_1}^{w_2} x^{\alpha} y^{\beta} z^{\gamma} w^{\delta}\, dx\, dy\, dz\, dw,$$

and the corresponding definite integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z}^{z_2}\int_{w_1}^{w_2} \mathrm{F}\left(\frac{x-1\,.\,z-1}{y-1\,.\,w-1}\right).\, x^{\alpha} y^{\beta} z^{\gamma} w^{\delta}\, dx\, dy\, dz\, dw$$

to the symbolic form

$$\mathrm{F}\left(\frac{\Delta_{\alpha}\,.\,\Delta_{\gamma}}{\Delta_{\beta}\,.\,\Delta_{\delta}}\right).\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}\int_{w_1}^{w_2} x^{\alpha} y^{\beta} z^{\gamma} w^{\delta}\, dx\, dy\, dz\, dw.$$

EXAMPLES.

(1.)

$$\int_{x_1}^{x_2} (\log x)^n \,.\, x^{a-1}\, dx = (D_a)^n \,.\, \frac{x_2{}^a - x_1{}^a}{a}.$$

Hence, as a special case,

$$\int_0^1 (\log x)^n \,.\, x^{a-1} dx = (D_a)^n \,.\, \left(\frac{1}{a}\right).$$

(2.)

$$\int_{x_1}^{x_2} \frac{x^{a-1} dx}{(x-1)^n} = \Delta_a{}^{-n} \,.\, \frac{x_2{}^a - x_1{}^a}{a} = \Sigma_a{}^n \,.\, \frac{x_2{}^a - x_1{}^a}{a}.$$

Hence, as a special case,

$$\int_0^1 \frac{x^{a-1} dx}{(x-1)^n} = \Sigma_a{}^n \,.\, \left(\frac{1}{a}\right).$$

(3.)

$$\int_{x_1}^{x_2} x^{p-1} (1-x)^{q-1} dx = (-1)^{q-1} \,.\, \Delta_p{}^{q-1} \,.\, \frac{x_2{}^p - x_1{}^p}{p}.$$

Hence, as a special case,

$$\int_0^1 x^{p-1} (1-x)^{q-1} dx = \mathsf{F}\,(p, q) = (-1)^{q-1} \,.\, \Delta_p{}^{q-1} \,.\, \left(\frac{1}{p}\right),$$

and, if q be any positive integer, we very easily derive for the value of the first Eulerian Integral, in this case,

$$\mathsf{F}\,(p,q) = \frac{1\,.\,2\,.\,3\,.\,.\,.\,.\,(q-1)}{p\,(p+1)\,(p+2)\,.\,.\,.\,(p+q-1)}.$$

(4.) Let it be proposed to reduce the integral, first discussed by Binet,

$$\int_0^1 \{(1+x)^{p-1}(1-x)^{q-1} + (1+x)^{q-1}(1-x)^{p-1}\}\, dx.$$

If, in our second fundamental theorem, we substitute $m+1$ for m, we obtain the theorem

$$\mathrm{F}\,(\Delta_a) \,.\, (1+x)^a = \mathrm{F}\,(x) \,.\, (1+x)^a,$$

a result, indeed, which could readily have been deduced directly.

Hence, it is easily seen that the integral proposed is equivalent to

$$(1-\Delta_p)^{q-1} \,.\, \int_0^1 (1+x)^{p-1}\, dx + (1-\Delta_q)^{p-1} \,.\, \int_0^1 (1+x)^{q-1} dx$$

or, finally,

$$(1-\Delta_p)^{q-1}\cdot\left(\frac{2^p-1}{p}\right)+(1-\Delta_q)^{p-1}\cdot\left(\frac{2^q-1}{q}\right).$$

For the evaluated result, Binet has obtained the very simple form

$$2^{p+q-1}\cdot \mathbf{F}\,(p,\,q).$$

(5.) To evaluate the integral

$$\int_0^1 (x^a-1)\,(1-x)^b dx$$

quoted by Haan from Lobatchewsky, *Mém. Kasan.* 1835.
It is obviously equivalent to

$$(-1)^b\Delta_a{}^b\cdot\left(\frac{1}{a+1}\right)+\int_0^1 (1-x)^b d\,(1-x),$$

or, at once, by a previous example,

$$\frac{1\,.\,2\,.\,3\,\ldots\ldots\,b}{(a+1)\,(a+2)\ldots(a+b)}-\frac{1}{b+1}.$$

The form in which the value of this integral is given by Haan is, in the notation of Kramp,

$$1^{b/1}\left\{\frac{1^{a/1}}{2^{a+b/1}}-\frac{1}{2^{b/1}}\right\};$$

the law of this notation being expressed by the formula

$$x^{m/n}=x\,(x+n)\,(x+2n)\ldots.(x+\overline{m-1}n).$$

(6.) It is a well-known theorem, due to M. Dirichlet, that if the variables x, y, z, &c., be connected by the condition

$$x+y+z+\&c.\ \angle 1,$$

then will the multiple definite integral

$$\iiint\ldots x^{l-1}y^{m-1}z^{n-1}\ldots dx\,dy\,dz\ldots=\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)\ldots}{\Gamma(1+l+m+n+\ldots)}.$$

Hence it follows that, if Φ be any algebraic function, the symbolic reduct form of the result of the evaluation of the multiple definite integral (proposed for discussion, Moigno, *Leçons de Cal. Diff. et Int.*, tome ii., p. 265),

$$\iiint\ldots\Phi\,(x+y+z+\&c.)\,.\,x^{l-1}y^{m-1}z^{n-1}\ldots dx\,dy\,dz\ldots$$

is simply

$$\Phi\left(e^{D_l}+e^{D_m}+e^{D_n}+\&c.\right).\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)\ldots}{\Gamma(1+l+m+n+\ldots)}.$$

As a simple case, if the variables x and y be connected by the condition

$$x+y \angle 1$$

then will

$$\iint \Phi(x+y)x^{l-1}y^{m-1}\,dx\,dy = \Phi\left(e^{D_l}+e^{D_m}\right).\frac{\Gamma(l).\Gamma(m)}{\Gamma(1+l+m)}.$$

It is obvious that, if the variables be connected by the same condition as before, and their number be p, we shall have for the result of the evaluation of the multiple definite integral

$$\iiint \ldots \Phi(x+y+z+\ldots-p)\,x^{l-1}y^{m-1}z^{n-1}\ldots dx\,dy\,dz\ldots,$$

in its symbolic reduct form, simply

$$\Phi(\Delta_l+\Delta_m+\Delta_n+\&c.) . \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)\ldots}{\Gamma(1+l+m+n+\ldots)}.$$

As a case of this last theorem, it appears that if the variables x and y be connected by the condition

$$x+y \angle 1,$$

then will

$$\iint \Phi(x+y-2)\,x^{l-1}y^{m-1}\,dx\,dy = \Phi(\Delta_l+\Delta_m).\frac{\Gamma(l).\Gamma(m)}{\Gamma(1+l+m)}.$$

There is no difficulty in extending these results to the more complicated case in which the variables are connected by the condition

$$\left(\frac{x}{a}\right)^p+\left(\frac{y}{\beta}\right)^q+\left(\frac{z}{\gamma}\right)^r+\&c. \overline{\underline{\angle}} 1.$$

(7.) It may readily be proved by the assumption $x^2=z$, that

$$\int_0^1 (1-x^2)^q x^{2a-1}\,dx = \tfrac{1}{2}.\frac{1.2.3\ldots.q}{a(a+1)(a+2)\ldots(a+q)},$$

or, in the notation of Kramp,

$$\tfrac{1}{2}.\frac{1^{q/1}.1^{a/1}}{a.1^{a+q/1}}.$$

In the same manner it may be proved that

$$\int_0^1 (1-x^2)^q x^{2a}\,dx = \frac{2.4.6\ldots.(2q)}{(2a+1)(2a+3)\ldots(2a+1+2q)},$$

or, in the notation of Kramp,

$$\frac{2^{q/2}}{(2a+1)^{q+1/2}}.$$

SECT. III.—*On certain Analogous Theorems in the Differential Calculus and the Calculus of Finite Differences.*

1. It is a well-known theorem in the operational or symbolic department of the Differential Calculus, that if Φ be any algebraic function of the symbol it contains, and u any subject,

$$e^{-mx} \,.\, \Phi(D) \,.\, e^{mx}u = \Phi(D+m)\,u.$$

By the aid of this principle, Dr. Boole has established his very remarkable theorem, that *the equation*

$$u + p_1\,\Phi(D)\,e^{x}u + p_2\,\Phi(D)\,\Phi(D-1)\,e^{2x}u + \dots \\ + p_n\,\Phi(D)\,\Phi(D-1)\,.\,.\,\Phi(D-n+1)\,e^{nx}u = U$$

may be resolved into a system of equations of the form

$$u - q\,\Phi(D)\,e^{x}u = U,$$

the values of q being determined by the equation

$$q^n + p_1\,q^{n-1} + p_2\,q^{n-2} + \dots + p_n = 0.$$

2. Similarly, in the Calculus of Finite Differences, we have the correlative primary theorems

$$a^{-x} \,.\, \mathrm{F}(e^D) \,.\, a^x u_x = \mathrm{F}(ae^D) \,.\, u_x,$$

and

$$a^{-x} \,.\, \mathrm{F}(\Delta) \,.\, a^x u_x = \mathrm{F}(a\Delta + \overline{a-1}) \,.\, u_x\,;$$

or, more generally,

$$a^{-mx} \,.\, \mathrm{F}(e^D) \,.\, a^{mx} u_x = \mathrm{F}(a^m e^D) \,.\, u_x,$$

and

$$a^{-mx} \,.\, \mathrm{F}(\Delta) \,.\, a^{mx} u_x = \mathrm{F}\,\{a^m e^D + (a^m - 1)\} \,.\, u_x\,;$$

each of which may be readily established inductively, and either of which may be deduced from the other.

The theorem in Finite Differences corresponding to that stated by Dr. Boole, and above quoted, is the following—(the reader being pleased to observe that the short symbolic general mode for writing an equation in Finite Differences, with constant coefficients, is

$$\Psi(e^D) \,.\, u_x = U)\text{—}$$

The equation

$$u_x + p_1\,\Phi(e^D)\,a^{mx}u_x + p_2\,\Phi(e^D)\,\Phi(a^{-m}e^D)\,a^{2mx}u_x + \dots \\ + p_n\,\Phi(e^D)\,\Phi(a^{-m}e^D)\,.\,.\,\Phi(a^{-(n-1)m}e^D)\,a^{nmx}u_x = U$$

may be resolved into a system of equations of the form

$$u_x - q\,\Phi\,(e^D)\,a^{mx}u_x = U,$$

the values of q being determined, as before, by the equation

$$q^n + p_1 q^{n-1} + p_2 q^{n-2} + \ldots + p_n = 0.$$

3. Again, as regards solution by series, if a differential equation, given to be solved, admitted of reduction to the typical form

$$F_o\,(xD)\,y + F_n\,(xD)\,x^n y = \Sigma A\,x^a, \tag{1}$$

where F_o, F_n are algebraic functions, and the order of the symbol xD is at least not lower in F_o than it is in F_n, we may proceed in the following manner.

Operating with the symbol $F_o\,(xD)^{-1}$ upon both sides, we get

$$y + \frac{F_n\,(xD)}{F_o\,(xD)} \cdot x^n y = \Sigma\,\frac{Ax^a}{F_o\,(a)} + \frac{1}{F_o\,(xD)} \cdot 0\,;$$

or, if the roots of the equation $F_o\,(p) = 0$ be supposed all real and unequal, and designated by α_1, α_2, α_3, &c.,

$$\left(1 + \frac{F_n\,(xD)}{F_o\,(xD)}\,x^n\right) y = \Sigma\,\frac{Ax^a}{F_o\,(a)} + \Sigma C x^{\alpha}.$$

Hence, at once,

$$y = \left\{ \begin{array}{c} \Sigma\,\dfrac{A}{F_o(a)}\left\{1 - \dfrac{F_n\,(xD)}{F_o\,(xD)}\,x^n + \dfrac{F_n\,(xD)}{F_o\,(xD)}\,x^n \cdot \dfrac{F_n\,(xD)}{F_o\,(xD)}\,x^n - \&c.\right\} x^a \\ + \\ \Sigma C\left\{1 - \dfrac{F_n\,(xD)}{F_o\,(xD)}\,x^n + \dfrac{F_n\,(xD)}{F_o\,(xD)}\,x^n \cdot \dfrac{F_n\,(xD)}{F_o\,(xD)}\,x^n - \&c.\right\} x^{\alpha}\,; \end{array} \right. \tag{2}$$

or, finally,

$$y = \left\{ \begin{array}{c} \Sigma\,\dfrac{Ax^a}{F_o(a)}\left\{1 - \dfrac{F_n\,(n+a)}{F_o\,(n+a)}\,x^n + \dfrac{F_n(2n+a)}{F_o(2n+a)} \cdot \dfrac{F_n(n+a)}{F_o(n+a)}\,x^{2n} - \&c.\right\} \\ + \\ \Sigma C\,x^{\alpha}\left\{1 - \dfrac{F_n\,(n+\alpha)}{F_o\,(n+\alpha)}\,x^n + \dfrac{F_n(2n+\alpha)}{F_o\,(2n+\alpha)} \cdot \dfrac{F_n\,(n+\alpha)}{F_o\,(n+\alpha)}\,x^{2n} - \&c.\right\}. \end{array} \right. \tag{3}$$

Let the roots of the equation $F_o\,(p) = 0$, be, as before,

$$a_1,\ a_2,\ a_3,\ \&c.,$$

and the roots of the equation $F_n(p) = 0$, be

$$\beta_1,\ \beta_2,\ \beta_3,\ \&c.$$

Then it is evident that if any one of the following relations, namely,

$$\begin{gathered} n + a = a_1,\ a_2,\ a_3,\ \&c., \\ 2n + a = a_1,\ a_2,\ a_3,\ \&c., \\ 3n + a = a_1,\ a_2,\ a_3,\ \&c., \\ \&c. \end{gathered}$$

should hold; or any one of the following,

$$\begin{gathered} n + a_1 = a_2,\ a_3,\ a_4,\ \&c.;\quad n + a_2 = a_1,\ a_3,\ \&c., \\ 2n + a_1 = a_2,\ a_3,\ a_4,\ \&c.;\ 2n + a_2 = a_1,\ a_3,\ \&c., \\ 3n + a_1 = a_2,\ a_3,\ a_4,\ \&c.;\ 3n + a_2 = a_1,\ a_3,\ \&c., \\ \&c., \end{gathered}$$

—in any of these cases, some of the terms in the above solution by series, become infinite, and therefore the corresponding equation ceases to be soluble by this method, if soluble at all.

On the other hand, if any one of the following relations, namely

$$\begin{gathered} n + a = \beta_1,\ \beta_2,\ \beta_3,\ \&c., \\ 2n + a = \beta_1,\ \beta_2,\ \beta_3,\ \&c., \\ 3n + a = \beta_1,\ \beta_2,\ \beta_3,\ \&c., \\ \&c., \end{gathered}$$

should hold; or any of the following:

$$\begin{gathered} n + a_1 = \beta_1,\ \beta_2,\ \beta_3,\ \&c.;\quad n + a_2 = \beta_1,\ \&c., \\ 2n + a_1 = \beta_1,\ \beta_2,\ \beta_3,\ \&c.;\ 2n + a_2 = \beta_1,\ \&c., \\ 3n + a_1 = \beta_1,\ \beta_2,\ \beta_3,\ \&c.;\ 3n + a_2 = \beta_1,\ \&c., \\ \&c., \qquad\qquad \&c.; \end{gathered}$$

then, in the former case, the series included within the first pair of brackets would become, in so far as the corresponding constituents are concerned, *terminate* (if the word may be employed in its participial sense); in the latter case, the series included within the second pair of brackets would become, to a like extent, terminate. It may be noticed here that when an equation is said to be integrable 'in finite terms,' the expression, as it stands, is somewhat ambiguous, but is intended to convey that the solution may be expressed in a determinate number of terms, as in contrast to an indefinite, though even converging series, or a series possibly reducible, but not yet reduced.

By this method may be treated the equation first proposed for solution by Euler, but, since his time, more completely identified with the name of another German mathematician. The equation alluded to is that known as Pfaff's differential equation, recently discussed by Dr. Boole in his valuable work on Differential Equations, and which, as remarked by this latter distinguished author, includes all examples of the second order, which are susceptible of reduction to the binomial type. It is of the form

$$(a + a'x^n)\, x^2 D^2 u + (b + b'x^n)\, xDu + (c + c'x^n)\, u = \mathrm{X}.$$

By the same method may be treated the still higher equation of the third order

$$(a + a'x^n)\, x^3 D^3 u + (b + b'x^n)\, x^2 D^2 u + (c + c'x^n)\, xDu + (d + d'x^n)\, u = \mathrm{X},$$

or, more generally, the differential equation of the r^{th} order,

$$(a + a'x^n)\, x^r D^r u + (b + b'x^n)\, x^{r-1} D^{r-1} u + \&c. + (k + k'x^n)\, u = \mathrm{X},$$

as is plain from observing that this last equation may be reduced to the form

$$(ax^r D^r + bx^{r-1} D^{r-1} + \&c. + k)\, u + x^n (a'x^r D^r + b'x^{r-1} D^{r-1} + \&c. + k')\, u = \mathrm{X},$$

or, transferring x^n to the right-hand side of the second operator,

$$\left.\begin{array}{r} \{axD\,(xD-1)\,..\,(xD-r+1) + bxD\,(xD-1)\,..\,(xD-r+2) + \&c.\}\; u \\ + \\ \{a'\,(xD-n)\,(xD-n-1)\,..\,(xD-\overline{n+r-1}) + b'\,(xD-n) \\ ..\,(xD-\overline{n+r-2}) + \&c.\}\; x^n u \end{array}\right\} = \mathrm{X},$$

which is obviously of the required form.

It will suffice to exhibit the application of the method to Pfaff's equation. This plainly can be reduced to the form

$$(xD - a_1)\,(xD - a_2)\, u + k\,(xD - \beta_1)\,(xD - \beta_2)\, x^n u = \Sigma \mathrm{M} x^m.$$

Hence at once we get

$$u + k\,\frac{(xD - \beta_1)\,(xD - \beta_2)}{(xD - a_1)\,(xD - a_2)}\, x^n u = \Sigma\, \frac{Mx^m}{(m - a_1)\,(m - a_2)} + A_1\, x^{a_1} + A_2\, x^{a_2},$$

or, finally, in full,

$$
u = \left\{ \begin{array}{l}
\Sigma \dfrac{Mx^m}{(m-\alpha_1)(m-\alpha_2)} \left\{ 1 - k \dfrac{(m+n-\beta_1)(m+n-\beta_2)}{(m+n-\alpha_1)(m+n-\alpha_2)} x^n \right. \\
\quad \left. + k^2 \dfrac{(m+2n-\beta_1)(m+2n-\beta_2)(m+n-\beta_1)(m+n-\beta_2)}{(m+2n-\alpha_1)(m+2n-\alpha_2)(m+n-\alpha_1)(m+n-\alpha_2)} x^{2n} - \&c \right\} \\
\qquad\qquad\qquad\qquad + \\
A_1 x^{\alpha_1} \left\{ 1 - k \dfrac{(\alpha_1+n-\beta_1)(\alpha_1+n-\beta_2)}{n(n+\alpha_1-\alpha_2)} x^n \right. \\
\quad \left. + k^2 \dfrac{(\alpha_1+2n-\beta_2)(\alpha_1+2n-\beta_2)(\alpha_1+n-\beta_1)(\alpha_1+n-\beta_2)}{2n^2(2n+\alpha_1-\alpha_2)(n+\alpha_1-\alpha_2)} x^{2n} - \&c. \right\} \\
\qquad\qquad\qquad\qquad + \\
A_2 x^{\alpha_2} \left\{ 1 - k \dfrac{(\alpha_2+n-\beta_1)(\alpha_2+n-\beta_2)}{n(n+\alpha_2-\alpha_1)} x^n \right. \\
\quad \left. + k^2 \dfrac{(\alpha_2+2n-\beta_1)(a^2+2n-\beta_2)(\alpha_2+n-\beta_1)(\alpha_2+n-\beta_2)}{2n^2(2n+\alpha_2-\alpha_1)(n+\alpha_2-\alpha_1)} x^{2n} - \&c. \right\}
\end{array} \right.
$$

It may be remarked that the same method of solution, precisely, applies to partial differential equations of the type

$$\mathrm{F}_o(xD_x + yD_y + zD_z + \&c.)\, u + \mathrm{F}_n(xD_x + yD_y + \&c.)\, \Theta_n u = \Sigma\Theta_m$$

where Θ_n, Θ_m are given homogeneous functions in the independent variables of the degrees n, m, respectively; or to the reducible type

$$\mathrm{F}_o(axD_x + byD_y + \&c.)\, u + \mathrm{F}_n(axD_x + byD_y + \&c.)\, \Theta_n u = \Sigma\Theta_m,$$

and that the partial differential equation, corresponding to the total differential equation discussed by Pfaff, namely

$$(a + a'\Theta_n)(x^2D^2_x + 2xyD_xD_y + y^2D^2_y)\, u + (b + b'\Theta_n)(xD_x + yD_y)\, u \\ + (c + c'\Theta_n)\, u = \Sigma\Theta_m,$$

where Θ_n is a given homogeneous function in x and y, admits of easy solution by a process similar to that already exhibited.

4. A similar method of solution will apply to the class of equations in Finite Differences represented by the typical form

$$\mathrm{F}_o(\Delta)\, u_x + \mathrm{F}_n(\Delta)\, e^{nx}\, u_x = \Sigma M e^{mx},$$

where, as before, F_o and F_n represent algebraic functions of the symbol Δ, and the degree of Δ in F_n is supposed to be at least not higher than in F_o. In fact, since

$$f(\Delta)\; Ca^x = f(a-1)\; Ca^x,$$

we have

$$u_x + \frac{F_n(\Delta)}{F_o(\Delta)} e^{nx} u_x = \Sigma \frac{Me^{mx}}{F_o(e^m-1)} + \frac{1}{F_o(\Delta)} 0;$$

or, if a_1, a_2, a_3, &c., be, as before, the roots of $F_o(p) = 0$,

$$u_x + \frac{F_n(\Delta)}{F_o(\Delta)} e^{nx} u_x = \Sigma \frac{Me^{mx}}{F_o(e^m-1)} + C_1(a_1+1)^x + C_2(a_2+1)^x + \&c.$$

Hence, at once,

$$u_x = \left\{ \begin{array}{c} \Sigma \dfrac{M}{F_o(e^m-1)} \left(1 - \dfrac{F_n(\Delta)}{F_o(\Delta)} e^{nx} + \dfrac{F_n(\Delta)}{F_o(\Delta)} e^{nx} \dfrac{F_n(\Delta)}{F_o(\Delta)} e^{nx} - \&c.\right) e^{mx}, \\ + \\ \Sigma C \left(1 - \dfrac{F_n(\Delta)}{F_o(\Delta)} e^{nx} + \dfrac{F_n(\Delta)}{F_o(\Delta)} e^{nx} \dfrac{F_n(\Delta)}{F_o(\Delta)} e^{nx} - \&c.\right) (a+1)^x, \end{array} \right.$$

or, finally,

$$u_x = \left\{ \begin{array}{l} \Sigma \dfrac{Me^{mx}}{F_o(e^m-1)} \left\{ 1 - \dfrac{F_n(e^{m+n}-1)}{F_o(e^{m+n}-1)} e^{nx} + \dfrac{F_n(e^{m+2n}-1) \cdot F_n(e^{m+n}-1)}{F_o(e^{m+2n}-1) \cdot F_o(e^{m+n}-1)} e^{2nx} - \&c. \right\} \\ \qquad + \\ \Sigma C (\alpha+1)^x \left\{ 1 - \dfrac{F_n(e^n \overline{\alpha+1} - 1)}{F_o(e^n \overline{\alpha+1} - 1)} e^{nx} + \dfrac{F_n(e^{2n} \overline{\alpha+1} - 1) \cdot F_n(e^n \overline{\alpha+1} - 1)}{F_o(e^{2n} \overline{\alpha+1} - 1) \cdot F_o(e^n \overline{\alpha+1} - 1)} e^{2nx} - \&c. \right\} \end{array} \right.$$

EXAMPLE.

Let it be proposed to solve the equation in finite differences, corresponding to Pfaff's differential equation

$$(a + a'p^x)\,\Delta^2 u_x + (b + b'p^x)\,\Delta u_x + (c + c'p^x)\,u_x = X = \Sigma M' e^{mx}.$$

This equation may obviously be thrown into the form

$$(\Delta - a_1)(\Delta - a_2)\,u_x + k\,(\Delta - \beta_1)(\Delta - \beta_2)\,p^x u_x = \Sigma M e^{mx}.$$

Hence we obtain the primary form—

$$\left\{1 + k\frac{(\Delta-\beta_1)(\Delta-\beta_2)}{(\Delta-a_1)(\Delta-a_2)} p^x\right\} u_x = \Sigma \frac{Me^{mx}}{(e^m-1-a_1)(e^m-1-a_2)} + C_1(1+a_1)^x + C_2(1+a_2)^x$$

and the solution, in series, required is—

$$u_x = \left\{ \begin{array}{c} \Sigma \dfrac{Me^{mx}}{(e^m - 1 - \alpha_1)(e_m - 1 - \alpha_2)} \left\{ 1 - k \dfrac{(pe^m - 1 - \beta_1)(pe^m - 1 - \beta_2)}{(pe^m - 1 - \alpha_1)(pe^m - 1 - \alpha_2)} p^x + \&c. \right\} \\ + \\ C_1 (1 + \alpha_1)^x \left\{ 1 - k \dfrac{(p\overline{1 + \alpha_1} - 1 - \beta_1)(p\overline{1 + \alpha_1} - 1 - \beta_2)}{(p - 1)(1 + \alpha_1)(p\overline{1 + \alpha_1} - \overline{1 + \alpha_2})} p^x + \&c. \right\} \\ + \\ C_2 (1 + \alpha_2)^x \left\{ 1 - k \dfrac{(p\overline{1 + \alpha_2} - 1 - \beta_1)(p\overline{1 + \alpha_2} - 1 - \beta_2)}{(p - 1)(1 + \alpha_2)(p\overline{1 + \alpha_2} - \overline{1 + \alpha_1})} p^x + \&c. \right\} \end{array} \right.$$

5. In the Differential Calculus, the following transformation of a theorem previously stated is occasionally useful, namely,

$$\frac{1}{x^m} . F(xD) . x^m u = F(xD + m) u.$$

The theorem in the calculus of Finite Differences, corresponding in expression, though not in power, to this, and possibly suggested before, may be thus enunciated. If

$$(x)_m = x(x + 1)(x + 2) \ldots (x + m - 1),$$

then will

$$\frac{1}{(x)_m} . F(x\Delta) . (x)_m u = F(\overline{x + m}\Delta + m) u,$$

a theorem which may be readily proved by induction from the fact that

$$x\Delta . (x)_m u = (x)_m (\overline{x + m}\Delta) m + u.$$

Mr. W. R. Wilde read a description of two ancient Irish Boats, found in the county of Cavan, which he presented to the Museum on the part of Alexander Nesbitt and Robert Burrowes, Esqrs.

The Ven. the Archdeacon of Ardfert read a paper "On an original Portrait of the Old Countess of Desmond."

Mr. Wilde presented the following donations to the Museum:—

A bronze dagger, found in the Dunshaughlin Crannoge, on the part of Mrs. Rothwell, of Kells.

An iron bridle-bit, on the part of Mrs. Tyrrell, of Ballinderry, near Enfield.

The thanks of the Academy were voted to the donors.